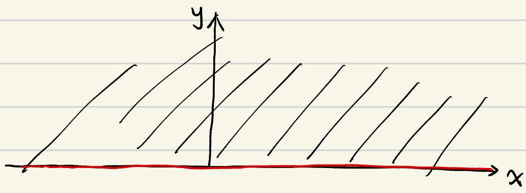
Fourier Analysis   April 11, 2024

Review.

Application 2: Steady state heat equation on the upper half plane.



$u = u(x,y)$ temperation distribution

$$(*) \quad \begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, & x \in \mathbb{R},\ y > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Set

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2+y^2}, \quad x \in \mathbb{R},\ y > 0.$$

By the method of Fourier transform, we derived that

$$u(x,y) = f * P_y(x), \qquad (x,y) \in \mathbb{R} \times \mathbb{R}_+$$

is a formal solution of $(*)$.

Next we will conduct a theoretic check.

Thm 1. Let $f \in S(\mathbb{R})$. Let

$$u(x,y) = f * P_y(x), \quad x \in \mathbb{R},\ y > 0.$$

Then u satisfies the following:

① $u \in C^2(\mathbb{R} \times \mathbb{R}_+)$ and $\Delta u = 0$

② $u(x,y) \rightrightarrows f(x)$ as $y \to 0$

③ $\int_{-\infty}^{\infty} |u(x,y) - f(x)|^2 dx \to 0$ as $y \to 0$

④ $u(x,y) \to 0$ as $|x| + y \to \infty$.

("u vanishes at ∞")

Pf. Here we only prove ④.

We will show that $\exists$ $C > 0$ such that

$$|u(x,y)| \le \begin{cases} C \cdot \left(\frac{1}{1+x^2} + \frac{y}{x^2+y^2} \right), \\ \frac{C}{y} \end{cases}.$$

for all $x \in \mathbb{R}$, $y > 0$

To this end, recall that

$$P_y(x) = \frac{1}{\pi}\,\frac{y}{x^2+y^2} \leq \frac{1}{\pi}\cdot\frac{1}{y}$$

Hence

$$|f * P_y(x)| \leq \int_{-\infty}^{\infty} |f(x-t)|\,|P_y(t)|\,dt$$

$$\leq \int_{-\infty}^{\infty} |f(x-t)|\cdot\frac{1}{\pi}\cdot\frac{1}{y}\,dt$$

$$= \frac{1}{\pi y}\cdot\int_{-\infty}^{\infty} |f(x)|\,dx$$

$$\leq \frac{C}{y}.$$

To see the other part.

Notice that

$$f * P_y(x) = \int_{-\infty}^{\infty} f(x-t)\,P_y(t)\,dt$$

$$= \int_{|t|\leq\frac{|x|}{2}} + \int_{|t|>\frac{|x|}{2}} f(x-t)\,P_y(t)\,dt$$

$$= (\mathrm{I}) + (\mathrm{II})$$

Then

$$|(\mathrm{I})| \leq \int_{|t| \leq \frac{|x|}{2}} |f(x-t)| \, P_y(t) \, dt$$

$$\left(|x-t| \geq \frac{|x|}{2} \right)$$

$$\leq \int_{|t| \leq \frac{|x|}{2}} \frac{C}{1+\left(\frac{|x|}{2}\right)^2} P_y(t) \, dt$$

$$\leq \frac{C}{1+\left(\frac{|x|}{2}\right)^2} \cdot \underbrace{\int_{-\infty}^{\infty} P_y(t) \, dt}_{=1}$$

$$\leq \frac{4C}{1+|x|^2}.$$

$$|(\mathrm{II})| \leq \int_{|t| \geq \frac{|x|}{2}} |f(x-t)| \cdot \frac{1}{\pi} \frac{y}{y^2+t^2} \, dt$$

$$\leq \int_{|t| \geq \frac{|x|}{2}} |f(x-t)| \cdot \frac{1}{\pi} \frac{y}{y^2+\frac{|x|^2}{4}} \, dt$$

$$\leq \frac{4y}{\pi(y^2+4x^2)} \cdot \int_{-\infty}^{\infty} |f(t)| \, dt$$

$$\leq \frac{\tilde{C} \, y}{y^2+x^2}.$$

$\blacksquare$

Thm 2 (Uniqueness)

Let $u \in C^2(\mathbb{R}\times\mathbb{R}_+) \cap C(\overline{\mathbb{R}\times\mathbb{R}_+})$.

Suppose u satisfies that

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R}\times\mathbb{R}_+ \\ u(x,0) = 0 & \text{on } \mathbb{R} \end{cases}$$

Moreover suppose that $u(x,y) \to 0$ as $|x|+y \to \infty$.

Then $u(x,y) \equiv 0$ on $\mathbb{R}\times\mathbb{R}_+$.

Remark: The condition that u "vanishes" at infinity can not be dropped.

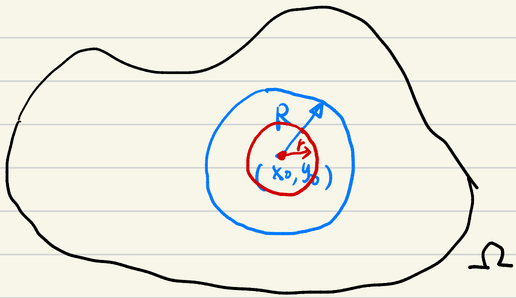
For instance, let $u(x,y) = y$. Then $\Delta u = 0$ and $u(x,0) = 0$.

Lemma 3 (Mean value property of harmonic function).

Let Ω be an open set of $\mathbb{R}^2$. Let $u \in C^2(\Omega)$ and suppose $\Delta u = 0$ on Ω. Suppose

$B_R(x_0, y_0) \subset \Omega$ where

$$B_R(x_0,y_0) = \{ (x,y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \le R^2 \}$$



Then $\forall \; 0 < r < R$,

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\theta, \, y_0 + r\sin\theta) \, d\theta$$

Below we prove Thm 2 by assuming Lemma 3. The proof of Lemma 3 will be postponed until next class.

Proof of Thm 1.

We use contradiction.

Suppose on the contrary that $u \not\equiv 0$.

Then $\exists\ (x_0, y_0) \in \mathbb{R} \times \mathbb{R}_+$ such that

$$u(x_0, y_0) \neq 0.$$

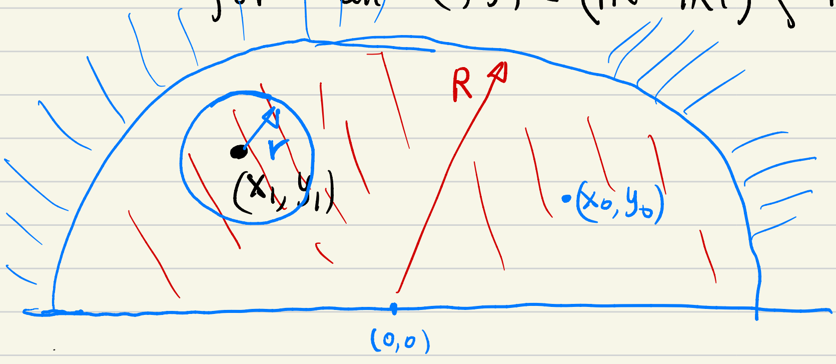
WLOG, we assume that $u(x_0, y_0) > 0$

(otherwise, we may consider $-u$)

Since u vanishes at infinity, we can find a large $R > 0$ such that

$$u(x, y) < \frac{1}{2} u(x_0, y_0)$$

for all $(x, y) \in (\mathbb{R} \times \mathbb{R}_+) \setminus B_R(0,0)$



Notice that u is cts on

$$B_R(0,0) \cap \overline{\mathbb{R}\times\mathbb{R}_+} \qquad (\text{closed, bdd})$$

So u attains its maximum on $B_R(0,0) \cap \overline{\mathbb{R}\times\mathbb{R}_+}$ at some (x_1, y_1)

i.e.

$$u(x_1, y_1) = \sup_{(x,y) \in B_R(0,0) \cap \overline{\mathbb{R}\times\mathbb{R}_+}} u(x,y)$$

Since $u(x,y) < \frac{u(x_0,y_0)}{2} < u(x_1,y_1)$ for $(x,y) \in \overline{\mathbb{R}\times\mathbb{R}_+} \setminus B_R(0,0)$, it follows that

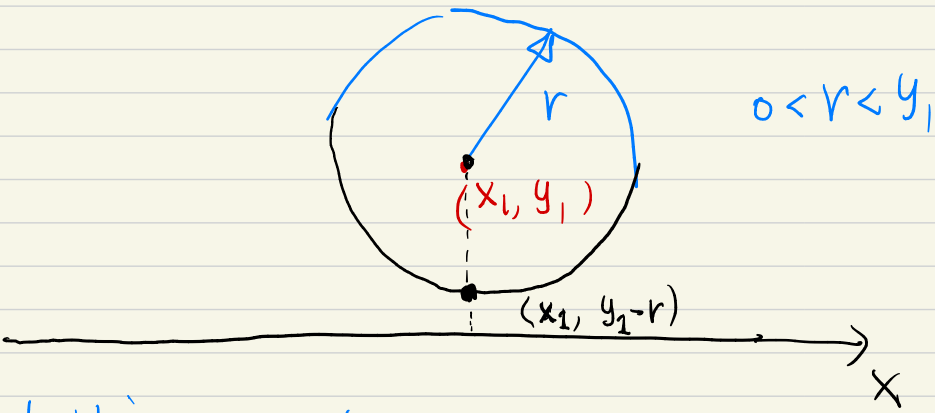
$$u(x_1, y_1) = \sup_{(x,y)\in \overline{\mathbb{R}\times\mathbb{R}_+}} u(x,y).$$

By the mean value property, for each

$$0 < r < y_1,$$

$$u(x_1, y_1) = \frac{1}{2\pi}\int_0^{2\pi} u(x_1 + r\cos\theta,\ y_1 + r\sin\theta)\, d\theta$$

$$\Rightarrow \quad u(x_1 + r\cos\theta,\ y_1 + r\sin\theta) = u(x_1, y_1)$$



Letting $r \to y_1$,

$$u(x_1, 0) = \lim_{r \to y_1} u(x_1, y_1 - r)$$

$$= u(x_1, y_1) > 0.$$

Contradiction! $\blacksquare$